APPLICATION OF CONJUGATE PROBLEMS OF HEAT CONDUCTION WITH A MOVING BOUNDARY TO THE ANALYSIS OF DRYING PROCESSES

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Inzhenerno-Fizicheskii Zhurnal, Vol. 14, No. 5, pp. 773–780, 1968

UDC 536.21:66.047.7

The conjugate problem of heat conduction with a moving boundary is formulated and solved in relation to conductive drying. The approximate solution of the problem is analyzed and compared with the experimental data.

The conductive drying of cellulose, paper, tar paper and other thin fibrous materials involves two stages: the periods of constant (first period) and falling (second period) drying rate [10]. The latter period is itself divided into two parts, the boundary between which is the second critical moisture content. Whereas the first period corresponds to constant or slowly falling average temperature, the first part of the second period is characterized by a sharply falling temperature, and the second part by a sharply increasing average temperature.

In accordance with the drying mechanism described by one of the authors, the second period begins with the depression of the evaporation zone below the heating surface. During the first part of the second period, this depression proceeds much more slowly than in the second.

Thus, in the second period the material is divided into two regions—a dry and a wet region; evaporation proceeds at the outer surface of the material and both at the boundary between the regions and in the moist material adjacent to the boundary. The vapor formed inside the material diffuses through the wet region to the free surface. The temperature of the evaporation surface at the boundary between the regions decreases with time. Whereas, in the first part of the second period, moisture is transported in the form of liquid and vapor, in the second part, it is transported primarily in vapor form.

In the second period, the amount of heat transferred to the material from the heating surface falls continuously with time.

It is important to make an analytical determination of the temperature fields in the drying material during the second period of conductive drying and compare them with the experimentally obtained fields. This would make it possible to refine the proposed mechanism of the process and develop a theoretical method of calculating conductive drying.

For this purpose, it is necessary to solve the nonsteady conjugate boundary value problem of heat conduction with a moving boundary between the dry and wet regions of the material. The dry region grows thicker and the wet region grows thinner with time.

As far as the authors are aware, such problems have not been solved or even applied for drying purposes except in the work of certain Japanese scientists [6] who attempted to analyze the drying process in relation to the end stage of convective drying of granular and powdered materials.

We attempt to formulate the problem as it applies to conductive drying.

Consider a moist capillary-porous body in the form of an infinite plate of thickness l (Fig. 1) in direct contact ($x = 0$) with a heating surface from which heat is transferred to the body exclusively by conduction at a flux density $q(\tau)$. The vapor formed can not be transported across the surface $x = 0$. At the free surface $x = l$, both the vapor formed at that surface and the vapor formed inside the material and transported through the body escape into the ambient vapor-air medium. The heat flow across an arbitrary surface in the body is composed of the heat transported by the skeletal structure of the body, by vapor, and by liquid.

The analytical solution of the conjugate problem of heat conduction for the first drying period with a steady boundary gives the initial temperature distribution over the thickness of the body $t_{1,2}(x, 0) = t_4(x, 0) = f_1(x)$, which is the initial distribution (at $\tau = 0$) for the second drying period. Consequently, the temperature of the body (t_W) in the plane $x = 0$ is known.

After time $\tau = 0$, the evaporation zone starts to be depressed and the body is divided into dry and wet regions, the position of the boundary being determined by the function $\xi(\tau)$. The value of this function gives

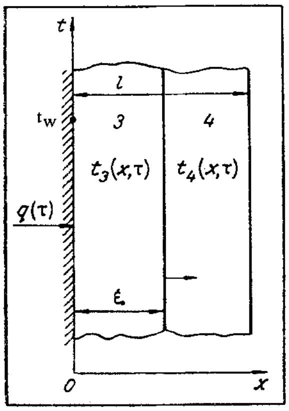

Fig. 1. Formulation of the problem.

the variable thickness of the dry region. The temperature at the boundary falls with time.

The region $0 \le x \le \xi(\tau)$, the dry region, consists of the capillary-porous skeleton of the body. The region $\xi(\tau) \le x \le l$, the wet region, consists of the skeleton of the body, liquid, and vapor; their temperatures are identical as a result of the intense heat transfer within the body. In view of the small changes in the thermophysical properties of the body within each part of the second period, the quantities a and λ may be assumed constant.

In capillary-porous bodies with a highly developed surface, evaporation takes place not only at the boundary but also in the adjacent wet volume. Therefore, heat sinks should be taken into account, both in the boundary condition at the dry-wet boundary and in the differential equation of the wet region, in the form of continuously distributed heat sinks.

However, for simplicity, we will assume that internal evaporation takes place only at the moving boundary, whose position is determined by the function $\xi(\tau)$.

The amount of heat expended on evaporation at the boundary can be determined from the total heat flux supplied during drying and the modified phase transition criterion ε [10]. This criterion estimates the fraction of heat transported by the vapor (formed, in this particular case, at the evaporation surface) relative to the total heat flux from the heating surface. It has different values in the first and second parts of the second period.

Mathematically, the problem may be formulated as follows:

$$\frac{\partial t_3}{\partial \tau} = a_3 \frac{\partial^2 t_3}{\partial x^2} \quad \text{at} \quad \tau > 0, \ 0 \leqslant x \leqslant \xi(\tau), \tag{1}$$

$$\frac{\partial t_4}{\partial \tau} = a_4 \frac{\partial^2 t_4}{\partial x^2} \quad \text{at} \ \tau > 0, \ \xi(\tau) \leqslant x \leqslant l = \text{const}. \tag{2}$$

The initial conditions are

$$t_3(x, 0) = t_4(x, 0) = f_1(x). \tag{3}$$

The boundary conditions:

a) at the outer fixed boundaries,

$$x = 0$$

$$-\lambda_3 \frac{\partial t_3(0, \tau)}{\partial x} = q(\tau), \quad q(\tau) \neq 0, \tag{4}$$

where $q(\tau)$ is some continuous function of time;

$$x = l$$

$$-\lambda_4 \frac{\partial t_4(l, \tau)}{\partial x} - (1 - \varepsilon) q(\tau) = 0; \tag{5}$$

b) at the moving interface at $x = \xi(\tau)$, the boundary conditions will be: equality of the temperatures of the dry and wet regions

$$t_3(\xi, \tau) = t_4(\xi, t) = t_w f(\tau), \tag{6}$$

where $f(\tau)$ is an arbitrary continuous function of time equal to 1 at $\tau = 0$, i.e., at the initial instant, the temperature at the interface is equal to t_w; equality of the heat fluxes with allowance for the phase-transition heat flux and the enthalpy change of the dried layer,

$$-\lambda_3 \frac{\partial t_3(\xi, \tau)}{\partial x} + \lambda_4 \frac{\partial t_4(\xi, \tau)}{\partial x} = \varepsilon r m(\tau) - \Delta I, \tag{7}$$

where $m(\tau)$ is some continuous function of time.

Equation (7) is the modified heat balance equation. The heat introduced at the interface together with the change of enthalpy due to phase transition (it "makes up" the heat flux) is expended on the evaporation of moisture at the moving boundary.

The quantity ΔI can be found from the expression

$$\Delta I = c_3 \rho_3 t_3(\xi, \tau) \frac{d\xi}{d\tau} - c_4 \rho_4 t_4(\xi, \tau) \frac{d\xi}{d\tau} =$$

$$= [c_3 \rho_3 t_3(\xi, \tau) - c_4 \rho_4 t_4(\xi, \tau)] \frac{d\xi}{d\tau}. \tag{8}$$

Substituting (8) into (7), we obtain

$$-\lambda_3 \frac{\partial t_3(\xi, \tau)}{\partial x} + \lambda_4 \frac{\partial t_4(\xi, \tau)}{\partial x} = m(\tau) r \varepsilon + [c_4 \rho_4 t_4(\xi, \tau) -$$

$$- c_3 \rho_3 t_3(\xi, \tau)] \frac{d\xi}{d\tau}. \tag{9}$$

The heat losses due to heat exchange with the ambient medium are small. Therefore, in condition (5) they are not taken into account.

We determine the temperature field in the dry and wet regions and the law of motion of the interface.

Problem (1)–(6), (9) differs from the classical Stefan problem [2–8] in that: a) the boundary conditions are functions of time; b) the temperature at the interface is not constant but varies with time; c) the boundary condition at the moving boundary also differs from the Stefan condition in that it contains time in explicit form; d) the problem is concerned with the temperature field in a body of finite dimensions. It is also different from the Verigin problem [9] and the Toei-Hayashi problem [6].

Since, with a minor exception, exact analytic solutions of problems with a moving boundary have not yet been obtained, we use an approximate method that yields a solution of engineering accuracy. This is Leibenzon's method [3] based on replacement of the actual temperature curves by their approximate analogs.

A mathematical analysis of the experimental data on the temperature fields in drying cellulose, cardboard, fiberglas, clay, and sand [1, 10] has shown that the t_3 and t_4 distributions with respect to the x-coordinate may, with a sufficient degree of accuracy, be assumed linear.

Then

$$t_3(x, \tau) = c_1(\tau) + c_2(\tau) x, \tag{10}$$

$$t_4(x, \tau) = c_3(\tau) + c_4(\tau) x. \tag{11}$$

In this case, the differential equations (1) and (2) are reduced to the form

$$a_3 \frac{\partial^2 t_3}{\partial x^2} = 0, \tag{12}$$

$$a_4 \frac{\partial^2 t_4}{\partial x^2} = 0. \tag{13}$$

From (12) and (13) and, correspondingly, from (10) and (11), respectively, we find that

$$\frac{\partial t_3}{\partial x} = c_2(\tau) \text{and } \frac{\partial t_4}{\partial x} = c_4(\tau).$$

Using boundary conditions (4) and (5), we obtain

$$c_2(\tau) = -\frac{1}{\lambda_3} q(\tau), \quad c_4(\tau) = -\frac{(1-\varepsilon)}{\lambda_4} q(\tau).$$

We find the constants $c_1(\tau)$ and $c_3(\tau)$, using boundary condition (6) and expressions (10) and (11):

$$c_1(\tau) = t_w f(\tau) + \frac{1}{\lambda_3} q(\tau)\xi, \quad c_3(\tau) = t_w f(\tau) + \frac{(1-\varepsilon)}{\lambda_4} q(\tau)\xi.$$

Substituting $c_1(\tau)$ and $c_2(\tau)$ into (10) and $c_3(\tau)$ and $c_4(\tau)$ into (11), we obtain the final expressions for the temperature distributions in the dry and wet regions of the body:

$$t_3(x, \tau) = t_w f(\tau) - \frac{1}{\lambda_3} q(\tau)[x - \xi], \tag{14}$$

$$t_4(x, \tau) = t_w f(\tau) - \frac{1-\varepsilon}{\lambda_4} q(\tau)[x - \xi]. \tag{15}$$

It follows from (15) that at $\tau = 0$, which corresponds to $\xi(0) = 0$,

$$t_4(x, 0) = f_1(x) = t_w - \frac{1-\varepsilon}{\lambda_4} q(0)x.$$

Thus, the initial condition must be a linear function of x, which corresponds to the experimental data.

Finding the expressions $\partial t_3(\xi, \tau)/\partial x$, $\partial t_4(\xi, \tau)/\partial x$, $t_3(\xi, \tau)$ and $t_4(\xi, \tau)$ from (14) and (15) and substituting them into Eq. (9), after transformations we obtain

$$\varepsilon q(\tau) - \varepsilon rm(\tau) = t_w f(\tau)[c_4\rho_4 - c_3\rho_3]\frac{d\xi}{d\tau}. \tag{16}$$

Integration of the latter differential equation gives

$$\xi = \int \frac{\varepsilon q(\tau) - \varepsilon rm(\tau)}{t_w f(\tau)[c_4\rho_4 - c_3\rho_3]} d\tau + C. \tag{17}$$

The constant of integration C can be found using the condition according to which $\xi = 0$ at $\tau = 0$.

Solutions (14), (15), and (17) have been used to calculate the process of conductive drying of cellulose (specific weight 0.3 kg/m^2) on a heating surface at a temperature of 130° C [1, 10].

The function $m(\tau)$ was determined from the drying rate [11]:

$$m(\tau) = -g\frac{d\bar{u}}{d\tau} = gk\bar{u}_{cr} \exp(-k\tau). \tag{18}$$

From an analysis of the heat-transfer data, obtained using the Rebinder number, it was found that

$$q(\tau) = q_0 \exp(-v\tau). \tag{19}$$

An analysis of the temperature fields obtained by various authors [10] has shown that the temperature at the wet-dry interface varies with time according to an exponential law:

$$f(\tau) = \exp(-h\tau). \tag{20}$$

Substituting (18), (19), and (20) into (17), integrating, and determining the arbitrary constant, we find the particular solution of Eq. (17),

$$\xi = \frac{1}{t_w[c_4\rho_4 - c_3\rho_3]}\left\{\frac{\varepsilon q_0}{v-h}[1 - \exp[-(v-h)\tau]] - \frac{B}{k-h}[1 - \exp[-(k-h)\tau]]\right\}, \tag{21}$$

where

$$B = \varepsilon rgk\bar{u}_{cr}. \tag{22}$$

Equation (21) expresses the law of depression of the evaporation zone during conductive drying and is the starting equation for determining ξ.

If we know the experimental dependence of ξ on τ, we can use Eq. (21) to calculate the phase-transition criterion ε.

The curve in Fig. 2 represents the depression of the evaporation zone for the experiment in question based on Eq. (21).

The calculation data corresponding to this experiment are: t_W = 102.5° C; q_0 = 16 550 W/m^2; $c_3\rho_3 = 2.46 \cdot 10^6$ J/m^3 · deg; $c_4\rho_4 = 3.39 \cdot 10^6$ J/m^3 · deg (first part of the period); $c_4\rho_4 = 3.16 \cdot 10^6$ J/m^3 · deg (second part of the period); $v = 1.695 \cdot 10^{-2}$ sec^{-1}; $h = 0.575 \cdot 10^2$ sec^{-1}; $k = 3.420 \cdot 10^2$ sec^{-1} (first part of the period); $k = 4.25 \cdot 10^2$ sec^{-1} (second part of the period); $\bar{u}_{cr_1}$ = 0.62 kg/kg; $\bar{u}_{cr_2}$ = 0.22 kg/kg; ε = 0.08 (first part of the period); ε = 0.20 (second part of the period).

The experimental points are plotted in the same figure. Clearly, these points coincide with the analytical curve. In the second part of the second period, the analytical curve given by Eq. (21) can be closely approximated by a linear function.

As for the first part of the second period, the experimental data are too few (only one point), and it is necessary to resort to the analytical dependence.

The rate of depression of the wet-dry interface increases slowly in the first part of the second period to a maximum that remains almost constant through the second part of the period.

From (14) and (16), using (18), (19), and (20) we calculated the temperature curves (Fig. 3) for the

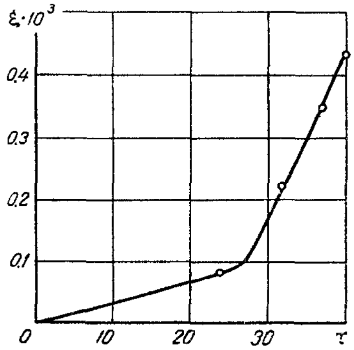

Fig. 2. Analytical curve showing the depression of the evaporation zone (ξ in m) with time (τ in sec) for conductive drying (experimental points indicated by circles).

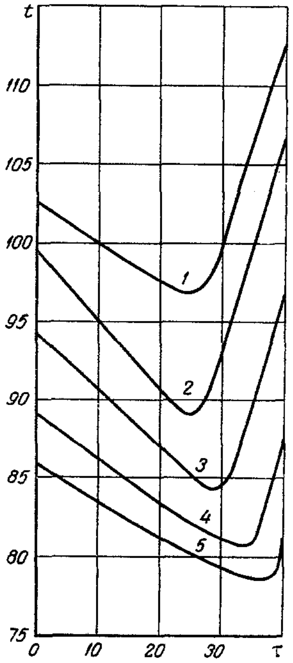

Fig. 3. Analytical curves showing the variation of temperature t (°C) in the layers with time τ (sec) in the second period of conductive drying of cellulose: 1) $x = 0$; 2) $8 \cdot 10^{-5}$ m; 3) $22 \cdot 10^{-5}$; 4) $35 \cdot 10^{-5}$; 5) $43 \cdot 10^{-5}$.

same experiment at $x = 0$; 0.08; 0.22; 0.35 and 0.43 mm, on the assumption that $\lambda_3 = 0.116$ W/m · deg; $\lambda_4 = 0.392$ W/m · deg (first part of the period); $\lambda_4 = 0.290$ W/m · deg (second part of the period).

The character of the analytical temperature curves corresponds to that of the experimental curves [1, 10]; from the start of the second period, the temperature of all the layers falls simultaneously; then, as drying proceeds, the temperature begins to rise. This temperature rise starts first in the layers nearest to the heating surface.

The correspondence of the analytic and experimental temperature curves proves the validity of our previous arguments [10] concerning the mechanism of conductive drying.

The calculated values of the temperatures t_4 are close to the experimental values (the maximum deviation is very small—2.5° C). The maximum deviation of the calculated and experimental values of the temperature t_3 for the layer $x = 0.08$ mm is 10° (at the 37-th second); for the other layers, the deviations are between 0 and 1.5° C.

For the 0.08-mm layer, the calculated values of t_4 are higher than the experimental values in the first part of the period; for the 0.43-mm layer, on the other hand, they are lower, while for the 0.22 and 0.35-mm layers they almost coincide. These regular deviations are a consequence of the somewhat schematized formulation of the problem, with evaporation assumed to take place only at the interface, whereas, in reality, internal evaporation extends over a certain volume of the material close to the interface, so that the evaporation at the surface is less intense.

The agreement between the analytical and experimental curves for the depression of the evaporation zone, and between the calculated and experimental values of the temperatures in the layers, together with the fact that the latter vary similarly with time, indicates that the problem has been correctly formulated and, hence, that the mechanism has been properly understood.

NOTATION

x is the coordinate, m; l is the thickness of the body, m; τ is the time, sec, t is the temperature of the body, °C; ξ is the moving coordinate of the wet-dry interface; λ is the thermal conductivity, W/m · deg; a is the thermal diffusivity, m^2/sec; ε is the modified phase-transition criterion; $q(\tau)$ is the density of the heat flux from the heating surface to the drying material (function of time), W/m^2; $f(\tau)$ is a function of time determining the law of temperature variation at the interface; $m(\tau)$ is the drying rate in the second period (function of time), kg/m^2 · sec; ΔI is the enthalpy change, J/kg; c is the specific heat, J/kg · deg; ρ is the density, kg/m^3; g is the specific weight, kg/m^2; k is the drying coefficient, sec^{-1}; $\bar{u}$ is the integral moisture content, kg/kg; v and h are parameters, sec^{-1}. Subscripts: w represents the surface of body with coordinate $x = 0$; 1 and 2 denote regions of the body in the first drying period; 3 and 4 represent dry and wet regions, respectively; cr indicates critical; 0 is initial.

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9 January 1968

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